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Supersymmetric models with product groups and field dependent gauge couplings

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Abstract: We study the dilaton-dependence of the effective action for $N = 1$, $SU(N_1) \times SU(N_2)$ models with one generation of vectorlike matter transforming in the fundamental of both groups. We treat in detail the confining and Coulomb phases of these models writing explicit expressions in many cases for the effective superpotential. We can do so for the Wilson superpotentials of the Coulomb phase when $N_2 = 2$, $N_1 = 2, 4$. In these cases the Coulomb phase involves a single $U(1)$ gauge multiplet, for which we exhibit the gauge coupling in terms of the modulus of an elliptic curve. The $SU(4) \times SU(2)$ model reproduces the weak-coupling limits in a nontrivial way. In the confining phase of all of these models, the dilaton superpotential has a runaway form, but in the Coulomb phase the dilaton enjoys flat directions. Had we used the standard moduli-space variables: $\mathrm{Tr}\,\mathcal{M}^k$, $k = 1, \cdots, N_2$, with $\mathcal{M}$ the quark condensate matrix, to parameterize the flat directions instead of the eigenvalues of $\mathcal{M}$, we would find physically unacceptable behaviour, illustrating the importance to correctly identify the moduli.

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Contents

1. Introduction

Great strides have been recently made in the understanding of nonperturbative effects in supersymmetric field theories [1, 2, 3, 4]: Seiberg and other workers have developed methods that allow us to write the exact form of the low-energy superpotential for many supersymmetric gauge theories. Using these methods, we explore the vacua and low-energy limit in a class of $N = 1$ supersymmetric gauge theories with gauge group $G = SU(N_1) \times SU(N_2)$.

We start by adding our motivations for the study of supersymmetric gauge theories with product gauge groups to those already given in ref. [5]. The strongest reason for their study is based on the following two observations. First, product gauge groups are ubiquitous in 'realistic' applications, including low-energy string models [6]. Second, their low-energy properties turn out to be interestingly different from those of supersymmetric theories involving simple gauge groups.

A feature of these models which is of particular interest to us is the dilaton dynamics which they predict. That is, in string theory the dilaton couples to low-energy gauge bosons in the following way:

$$\mathcal{L}_{\rm kin} = \frac{1}{4} \sum_{r=1}^{2} \Big(S_r \,{\rm Tr}\, \mathcal{W}_r \mathcal{W}_r \Big)_F , \tag{1.1}$$

where $\Big(\cdots \Big)_F$ denotes the chiral supersymmetric invariant, $\mathcal{W}_r$ is the gauge-kinetic chiral spinor supermultiplet, and the S_r's are related to the dilaton superfield, S, by the well-known relation

$$S_r = k_r \, S \,. \tag{1.2}$$

Here the k_r are the Kac-Moody levels of the corresponding gauge-group factors. Moduli dependence due to threshold effects, or extra contributions to S_r due to nonperturbative string dynamics can also be considered.

The low-energy scalar potential for the dilaton is important because eq. (1.1) implies the vacuum values of the scalar components, s_r, of the superfields S_r, play the role of the low-energy gauge coupling, g_r, being related to these couplings and the vacuum angles, Θ_r, by:

$$s_r = \frac{1}{g_r^2} - \frac{i\Theta_r}{8\pi^2} \,. \tag{1.3}$$

Moreover, within string theory general nonrenormalization theorems imply that the dilaton scalar potential is not generated in perturbation theory, and so its determination is a strong-coupling problem.

Some generic, and troubling, features of the low-energy dilaton superpotential, $W(S)$, in four-dimensional supersymmetric string vacua have emerged over the past decade and a half of study. After elimination of other low-energy fields, these superpotentials typically have the form [7]:

$$W(S) = \sum_k A_k e^{-a_k \, S} , \tag{1.4}$$

where A_k and a_k are numbers which depend on the model considered, with a_k generally positive.[1] Such a superpotential implies a scalar potential with a runaway minimum, at $\Re s \to \infty$.

In fact, the existence of this kind of runaway solution appears to be model independent, as may be seen from eq. (1.3). Since the dilaton plays the role of the gauge coupling, and since flat space with zero coupling is a well-known string vacuum, the scalar potential for the low-energy modes of four-dimensional string solutions generally

[1] A class of models for which some of the a_k are negative have recently been constructed in ref. [8], using nonabelian gauge groups which are not asymptotically-free.

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lead to scalar potentials for which the dilaton is driven towards zero coupling: $\Re s \to \infty$ [9]. Such runaway behaviour follows quite generally so long as these models are continuous in the zero-coupling limit. (It is this last continuity assumption which is evaded in the models of ref. [8].)

The low-energy dilaton dynamics which we find for the product groups explored here is as follows. The models exhibit several phases, and the low-energy degrees of freedom which arise depend on which phase is involved. The models typically exhibit a confining phase, for which the nonabelian gauge dynamics are confined, with a gap between the strongly-coupled ground state and its low-energy excitations. In this phase we find eq. (1.4) applies, leading to the usual dilaton runaway.

There is also another, Coulomb, phase which involves massless degrees of freedom, and in this phase we find a low-energy (Wilson) superpotential having flat directions for the dilaton field, even after the strong-coupling effects are included. This phase therefore differs from eq. (1.4) inasmuch as the dilaton is not *driven* to infinity by strong-interaction effects. What value its v.e.v. ultimately takes cannot be determined without more information, in particular as to how supersymmetry is ultimately spontaneously broken.

The models we consider have gauge group $G = SU(N_1) \times SU(N_2)$, with $N_1 \geq N_2 \geq 2$. We choose matter which transforms only in the fundamental representation of both of the factors of the gauge group, $R = (\mathbf{N_1}, \mathbf{N_2}) \oplus (\overline{\mathbf{N}}_\mathbf{1}, \overline{\mathbf{N}}_\mathbf{2})$. Although the inclusion of field-dependent gauge couplings for product gauge groups is not novel in itself, earlier workers have not done so for matter field carrying charges for more than one factor of a product gauge group [10, 11]. Models having the gauge group $G = SU(N_1) \times SU(N_2)$ have also been examined by other authors [3, 5, 12, 13, 14, 15], although with a matter content which differs from what we consider here. The special case where $N_1 = N_2 = 2$ is also analyzed in ref. [16].

We present our results in the following way. Section 2 starts with the construction of the effective superpotential for the factor-group models we wish to explore. After presenting some preliminaries, we state the general symmetries and limiting behaviour which guide the determination of the model's effective superpotential. Because their low-energy behaviour differs dramatically, we consider separately the cases where the mass of the quark supermultiplets is zero (the Coulomb phase) and nonzero (the confining phase).

We explore our first application of the general results in section 3, where we solve in explicit detail for the superpotential and gauge coupling function of a simple illustrative model, consisting of one generation of matter transforming as a $(\mathbf{2}, \mathbf{2}) \oplus (\overline{\mathbf{2}}, \overline{\mathbf{2}})$ of the gauge group $SU(2) \times SU(2)$. We argue that the Coulomb phase of this model has a low-energy superpotential which is completely dilaton-independent, evading the problem of the runaway dilaton by making the dilaton a *bona fide* flat direction, even after the inclusion of nonperturbative quantum effects. Our results for this particular model reproduce those of ref. [16].

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Next, section 4 presents another simple model, consisting of one generation of matter transforming as a $(\mathbf{4},\mathbf{2}) \oplus (\overline{\mathbf{4}},\overline{\mathbf{2}})$ of the gauge group $SU(4) \times SU(2)$. The weak-coupling limit of this model has interesting complications because its low-energy spectrum changes fundamentally in the limit of vanishing $SU(4)$ coupling. We are led in a different way to a similar conclusion as for the model of section 3: to a low-energy superpotential with directions along which the dilaton can vary without breaking supersymmetry, and so with no cost in energy. We also present ansätze for the gauge-coupling functions for the Coulomb phases of the models of section 3 and section 4.

Finally, section 5 briefly summarizes our conclusions.

2. $SU(N_1) \times SU(N_2)$ models

We now collect results which apply generally to models having gauge group $SU(N_1) \times SU(N_2)$, with one generation of matter fields:

$$Q_{a\alpha} \in (\mathbf{N_1}, \mathbf{N_2}) \qquad \text{and} \qquad \widetilde{Q}^{a\alpha} \in (\overline{\mathbf{N}}_\mathbf{1}, \overline{\mathbf{N}}_\mathbf{2})\,. \tag{2.1}$$

We use here $a, b, c, \ldots$ as the gauge indices of $SU(N_1)$, and $\alpha, \beta, \gamma, \ldots$ as those of $SU(N_2)$. We may take, without loss of generality, $N_1 \geq N_2 \geq 2$. Finally, we assume the microscopic superpotential to involve only a quark mass term:

$$w(Q, \widetilde{Q}) = m\; Q_{a\alpha} \widetilde{Q}^{a\alpha}\,. \tag{2.2}$$

Except for the special case $N_1 = N_2 = 3$, this is the only term possible which is both renormalizable and gauge invariant.

Our goal is to construct the effective superpotential and, where relevant, the effective gauge couplings of this model. We do so following what have become standard methods, and those readers interested in the applications to the $SU(2)\times SU(2)$ and $SU(4)\times SU(2)$ models can proceed directly to section 3 and section 4. Our notation and the details of our procedure follow those of ref. [18], (see also [17]) and we also temporarily maintain the fiction that the quantities S_r are independent fields, with the connection to the single dilaton field, S, through eq. (1.2), deferred to the final expressions.

2.1. The semiclassical spectrum

We start by sketching the low-energy phases which are indicated semiclassically, directly using the microscopic degrees of freedom. These are determined by examining the minima of the classical scalar potential,

$$V = \left|\mathcal{F}_{a\alpha}\right|^2 + \left|\widetilde{\mathcal{F}}^{a\alpha}\right|^2 + \frac{1}{2}\, D_A^2\,. \tag{2.3}$$

where $\mathcal{F}_{a\alpha} = \left(\partial w/\partial \mathcal{Q}_{a\alpha}\right)^*$, $\widetilde{\mathcal{F}}^{a\alpha} = \left(\partial w/\partial \widetilde{\mathcal{Q}}^{a\alpha}\right)^*$ and $D_A = \mathcal{Q}^\dagger T_A \mathcal{Q} + \widetilde{\mathcal{Q}}^\dagger \widetilde{T}_A \widetilde{\mathcal{Q}}$. Here $\mathcal{Q}_{a\alpha}$ and $\widetilde{\mathcal{Q}}^{a\alpha}$ represent the scalar components of the superfields $Q_{a\alpha}$ and $\widetilde{Q}^{a\alpha}$, while

T_A and $\widetilde{T}_A$ represent the generators of the gauge group acting on these fields. Finally, w represents the microscopic superpotential, given by eq. (2.2).

Clearly the scalar potential differs qualitatively according to whether or not the quark masses satisfy $m = 0$, since if this is true the superpotential w identically vanishes. We therefore present the semiclassical analysis separately for these two cases.

- *The Coulomb phase ($m = 0$):*

  Consider first the case $m = 0$, for which the microscopic superpotential vanishes. In this case the classical scalar potential vanishes along any scalar field configuration for which the D_A vanish. These conditions define the following D-flat ($D^\flat$) directions:

$$\mathcal{Q}_{a\alpha} = \begin{pmatrix} v_1 & & \\ & \ddots & \\ & & v_{N_2} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \qquad \text{and} \qquad \widetilde{\mathcal{Q}}^{a\alpha} = \begin{pmatrix} \tilde{v}_1 & & \\ & \ddots & \\ & & \tilde{v}_{N_2} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \tag{2.4}$$

  provided the nonzero coefficients satisfy $v_i = \tilde{v}_i^*$, for each $i = 1, \dots, N_2$. These field configurations do not break supersymmetry, but for generic values the gauge group is broken down to a subgroup, H. Three cases arise, each having a different H:

  1. If $N_1 = N_2$ the unbroken subgroup is simply $H = [U(1)]^{N_2 - 1}$, where each of the $U(1)$ factors is in a diagonal subgroup of the two gauge-group factors.
  2. If $N_1 = N_2 + 1$ the unbroken subgroup becomes $H = [U(1)]^{N_2}$, where the additional $U(1)$ factor corresponds to phase rotations of the bottom row of $\mathcal{Q}_{a\alpha}$ and $\widetilde{\mathcal{Q}}_{a\alpha}$.
  3. If $N_1 \geq N_2 + 2$ the unbroken subgroup is $H = SU(N_1 - N_2) \times [U(1)]^{N_2 - 1}$.

  For special values of v_i and $\tilde{v}_i$ the unbroken semiclassical symmetry group can be larger than this.

  We are therefore led, semiclassically for $m = 0$, to a Coulomb phase in which the low-energy theory is supersymmetric, containing several $U(1)$ gauge multiplets, plus a number of matter multiplets which parameterize the potential's $D^\flat$ directions.[2] (Any nonabelian factors of the gauge group are expected to confine, and so to drop out of the very-low-energy sector.)

  The number of complex fields required to parameterize these semiclassical flat directions [19, 20] is the total number of complex scalar fields, $\mathcal{S} = 2N_1N_2$, less

[2]We thank Eric Poppitz for identifying an error in our previous treatment of the $D^\flat$ directions.

the number of broken generators of the gauge group, $\mathcal{B} = \dim(G/H)$. The superpotential of the Wilson action for these low-energy degrees of freedom therefore requires $\mathcal{D} = \mathcal{S} - \mathcal{B}$ matter fields as its arguments, describing these $D^\flat$ directions.

For each of the three cases for H considered above we therefore find:

1. If $N_1 = N_2$ then $H = [U(1)]^{N_2-1}$, so the semiclassical low-energy spectrum contains $(N_2 - 1)$ $U(1)$ gauge supermultiplets and $\mathcal{D} = 2N_2^2 - [2(N_2^2-1) - (N_2-1)] = N_2 + 1$ $D^\flat$ directions.
2. If $N_1 = N_2 + 1$ then $H = [U(1)]^{N_2}$, so the semiclassical low-energy spectrum contains N_2 $U(1)$ gauge multiplets and $\mathcal{D} = 2(N_2+1)N_2 - \Big[\big((N_2+1)^2 - 1\big) + (N_2^2 - 1) - N_2\Big] = N_2 + 1$ $D^\flat$ directions.
3. If $N_1 \geq N_2 + 2$ then $H = SU(N_1 - N_2) \times [U(1)]^{N_2-1}$, so the semiclassical low-energy spectrum contains $(N_2 - 1)$ $U(1)$ gauge supermultiplets and $\mathcal{D} = 2N_1N_2 - \Big[(N_1^2-1) + (N_2^2-1) - \big((N_1-N_2)^2 - 1\big) - (N_2-1)\Big] = N_2$ $D^\flat$ directions. As mentioned earlier, the nonabelian $SU(N_1 - N_2)$ gauge multiplet is expected to confine and so to decouple from the low-energy theory.

- *The confining phase ($m \neq 0$):* If $m \neq 0$, then the degeneracy along the $D^\flat$ directions is directly lifted, even semiclassically, by the microscopic superpotential, eq. (2.2), indicating that the squark fields vanish in the vacuum. In this case the semiclassical massless spectrum therefore consists of a nonabelian $SU(N_1) \times SU(N_2)$ supersymmetric gauge multiplet, with no massless matter multiplets. Keeping in mind that the gauge multiplets are expected to confine, we therefore expect in this case a gapped low-energy theory with no massless states.

In the following sections we explore these two phases in considerably more detail.

2.2. Which effective superpotential?

There are two kinds of superpotentials which are useful for exploring the vacuum and low-energy properties of these (and other) supersymmetric gauge theories. On the one hand there is the superpotential for the 'exact quantum effective action', which generates the irreducible correlation functions of the theory. The arguments of this superpotential can be chosen to be any fields whose correlations are to be studied. The other superpotential is that for the 'Wilson' action which describes the dynamics of the theory's low-energy modes.

For the present purposes the following properties of these superpotentials are the most important:[3]

[3]See ref. [18] for more details concerning the definitions and differences between these superpotentials within the context of supersymmetric gauge theories.

- *(1) Locality:* Because the Wilson action receives no contributions from massless states, it is guaranteed to be a local quantity. This property is crucial, since it underlies the holomorphy of the superpotential which determines the vacuum properties [1]. The same need *not* be true for the quantum action if the system involves massless degrees of freedom.

- *(2) Linearity:* As is proven in [18], if the arguments of the quantum action are taken to include the variables

$$M \equiv \left\langle Q_{a\alpha}\tilde{Q}^{a\alpha}\right\rangle, \qquad \text{and} \qquad U_r \equiv \left\langle \mathrm{Tr}\, \mathcal{W}_r \mathcal{W}_r \right\rangle, \quad r = 1,2 \tag{2.5}$$

  then the definitions imply the conjugate quantities, m and S_r, can appear in the superpotential only through the terms mM and $\frac{1}{4}\sum_r S_r U_r$.

- *(3) Equivalence:* For systems having no massless degrees of freedom, it can happen that the quantum action coincides with the Wilson action if their arguments are chosen to be the same fields. This is because both are local due to the absence of massless states, and the symmetries of the problem may then uniquely determine the form of the result.

Which of these actions is relevant depends on the question of physical interest, and on which of the theory's phases is under consideration. For example, for the Coulomb phase ($m = 0$) it is the effective superpotential and gauge-coupling function for 'the' Wilson action governing the dynamics of the massless modes which we construct. (The word 'the' appears in quotations here because in reality there is potentially a different Wilson action for each vacuum of the model.) For the confining phase, on the other hand, it is the superpotential for both the Wilson and quantum effective actions which we compute. Both are local because of the absence of gapless modes, and the symmetries of the model force them to be identical when evaluated for appropriately chosen field configurations.

2.3. Global symmetries

In order to determine the form of the superpotential of the effective theories we take advantage of the global symmetry group which the model enjoys when $m = 0$. (This would also be a symmetry when $m \neq 0$, provided we also transform m appropriately.) For generic values of N_1 and N_2, this symmetry group is $U_A(1) \times U_B(1) \times U_R(1)$, defined by:

$$\begin{aligned} Q_{a\alpha}(\theta) &\to e^{i\beta_A + i\beta_B + 2i\beta_R/3} Q_{a\alpha}\left(e^{i\beta_R}\theta\right), \\ \tilde{Q}^{a\alpha}(\theta) &\to e^{i\beta_A - i\beta_B + 2i\beta_R/3} \tilde{Q}^{a\alpha}\left(e^{i\beta_R}\theta\right), \\ \mathcal{W}_r(\theta) &\to e^{i\beta_R} \mathcal{W}_r\left(e^{i\beta_R}\theta\right), \end{aligned} \tag{2.6}$$

where β_A, β_B and β_R are the transformation parameters.

The effective superpotential is constructed by requiring it to realize these symmetries in the same way as does the microscopic theory. For instance, if the arguments of the superpotential are the variables M and U_r of eq. (2.5), then the action of the global symmetry follows from eqs. (2.5) and (2.6):

$$M(\theta) \to e^{2i\beta_A + 4i\beta_R/3} M\left(e^{i\beta_R}\theta\right) \qquad U_r(\theta) \to e^{2i\beta_R} U_r\left(e^{i\beta_R}\theta\right). \tag{2.7}$$

The anomaly-free symmetry, $U_B(1)$, must simply be a symmetry of $W(U_r, S_r, M)$. For the anomalous $U_A(1) \times U_R(1)$ transformations, however, W must reflect the microscopic theory's property that shifts of the S_r are required to cancel the anomalies. Since the various symmetries have separate anomalies with each of the gauge group factors — all mixed anomalies vanish which involve both gauge groups simultaneously — independent shifts are required for each of the superfields S_r. These are possible so long as these fields are regarded as being independent of one another.

The upshot is that the effective action must be invariant with respect to the anomalous symmetries, provided that eq. (2.7) is supplemented by an appropriate transformation law for S_r. The required transformation is simply formulated in terms of the fields $L_r \equiv \exp[-4\pi^2 S_r]$, for which:

$$L_r(\theta) \to e^{i\mathcal{A}^r_A\beta_A + i\mathcal{A}^r_R\beta_R}\, L_r\left(e^{i\beta_R}\theta\right), \qquad \text{with} \qquad \mathcal{A}^r_X = \sum_i T(\mathcal{R}^r_i) Q_X(\mathcal{R}^r_i). \tag{2.8}$$

Here $X = A, R$ distinguishes the two anomalous symmetries, and $\sum_i$ is a sum over the gauge representations, $\mathcal{R}^r_i$, of all of the left-handed spin-half fields of the model. $Q_X(\mathcal{R}^r_i)$ denotes the quantum number of these fields under the anomalous symmetry $X = A, R$. $T(\mathcal{R}^r_i)$ is defined in terms of the trace of the gauge generators in the representation of interest, *via* $\mathrm{Tr}_{\mathcal{R}^r}[t_a t_b] \equiv T(\mathcal{R}^r)\, \delta_{ab}$. We use the standard convention for which the gauge generators are normalized so that $T(F) = \frac{1}{2}$ in the fundamental representation, and so then $T(A) = N_c$ for the adjoint representation of $SU(N_c)$. For instance, when evaluated for supersymmetric QCD (SQCD) with N_f quark flavours, the quarks give $\sum_i T(\mathcal{R}_i) = N_f$, while for the gauginos we have $T(A) = N_c$.

For the product model of interest the coefficients $\mathcal{A}^r_X$, appearing in eq. (2.8), become:

$$\begin{aligned} \mathcal{A}^1_A &= N_2\,, & \mathcal{A}^2_A &= N_1\,, \\ \mathcal{A}^1_R &= N_1 - \frac{N_2}{3}\,, & \mathcal{A}^2_R &= N_2 - \frac{N_1}{3}\,. \end{aligned} \tag{2.9}$$

Before determining the implications for the effective superpotential of these transformation rules, we pause in passing to record the discrete subgroup of $U_A(1) \times U_R(1)$ which is anomaly-free. This is most simply determined by requiring that the vacuum angle, Θ_r, for each of the gauge-group factors to become shifted by an integer multiple of 2π. Keeping in mind the relationship, (1.3), between the scalar part of the S_r and the gauge couplings, g_r and the vacuum angle, Θ_r we see that it is the field L_r^2 which has argument Θ_r. The anomaly-free discrete symmetry subgroup is therefore defined by

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the conditions: $L_r^2 \to e^{2\pi i n_r}\, L_r^2$, where n_1 and n_2 are integers. Requiring eqs. (2.8) and (2.9) to have this effect implies the following solutions for the allowed transformation parameters:

$$\begin{aligned} \frac{\beta_A}{2\pi} &= \frac{\left(N_1 - \frac{1}{3}\, N_2\right)\, n_2 - \left(N_2 - \frac{1}{3} N_1\right)\, n_1}{2(N_1^2 - N_2^2)}\,, \\ \frac{\beta_R}{2\pi} &= \frac{N_1\, n_1 - N_2\, n_2}{2(N_1^2 - N_2^2)}\,. \end{aligned} \tag{2.10}$$

This same result can also be obtained by counting the zero modes appearing in nonzero instanton amplitudes.

2.4. Limiting cases

Besides being constrained by these symmetries, the effective quantum action and Wilson action of the model are also subject to boundary conditions, as the parameters s_1, s_2 and m take various special values. This section outlines these boundary conditions.

- *(I) $m \to \infty$:* In the limit of large m the quark supermultiplets of the microscopic theory must decouple, leaving the theory of the pure gauge supermultiplet for the gauge group $SU(N_1) \times SU(N_2)$, with no matter. The superpotential for the quantum effective action for this theory is well known, being simply the sum of the result for each of the separate gauge factors.

- *(II) $\Re s_2 \to \infty$:* When the gauge coupling of the $SU(N_2)$ factor is taken to zero we are left with supersymmetric QCD (SQCD), with $N_c = N_1$ colours and $N_f = N_2$ flavours. The global symmetry group (for $m = 0$) in this case is larger than for finite s_2 because the absence of $SU(N_2)$ gauge interactions implies we are free to rotate the fields $Q_{a\alpha}$ and $\widetilde{Q}^{a\alpha}$ independently of one another. The flavour symmetry therefore in this limit becomes $G_f = SU(N_2) \times \widetilde{SU}(N_2) \times U_A(1) \times U_B(1) \times U_R(1)$.

- *(III) $\Re s_1 \to \infty$:* When the gauge coupling of the $SU(N_1)$ factor is taken to zero we again have SQCD, this time with $N_c = N_2$ colours and $N_f = N_1$ flavours. The global symmetry group (for $m = 0$) in this case is therefore $G_f = SU(N_1) \times \widetilde{SU}(N_1) \times U_A(1) \times U_B(1) \times U_R(1)$.

  In the special case $N_2 = 2$ there is a still larger global flavour symmetry because the gauge representations $\mathbf{2}$ and $\overline{\mathbf{2}}$ are equivalent to one another. In this case the flavour group becomes $G_f = SU(2N_1) \times U_A(1) \times U_R(1)$.

2.5. The effective superpotential in the confining phase

Consider first the confining phase ($m \neq 0$) for which we wish to compute the superpotential for the effective quantum action. For simplicity we consider as arguments for this action simply the variables M and U_r of eq. (2.5). Our goal is to demonstrate that the runaway dilaton superpotential *is* generic for the confining phase of the product models.

The global $U(1)$ symmetries, together with the exact linearity requirement that S_r appear only in the term $\frac{1}{4}\sum_r U_r S_r$, and the quark mass appear only through the term mM, determine the superpotential to have the following form:

$$\begin{aligned} W &= \frac{1}{32\pi^2}\sum_{r=1}^{2} U_r \log\left(\frac{U_r^{a_r} M^{b_r}}{L_r^2\, \mu_r^{3a_r+2b_r}}\right) + w\left(U_1, U_2\right) + mM \\ &= \frac{1}{4}U_1 S_1 + \frac{U_1}{32\pi^2}\left[a_1 \log\left(\frac{U_1}{\mu_1^3}\right) + b_1 \log\left(\frac{M}{\mu_1^2}\right)\right] \\ &\quad + \frac{1}{4}U_2 S_2 + \frac{U_2}{32\pi^2}\left[a_2 \log\left(\frac{U_2}{\mu_2^3}\right) + b_2 \log\left(\frac{M}{\mu_2^2}\right)\right] + w\left(U_1, U_2\right) + mM\,. \end{aligned} \tag{2.11}$$

Here the function $w(x, y)$ which appears in eq. (2.11) is completely arbitrary, subject only to the symmetry requirement that it be homogeneous of degree one, *i.e.*: $w(\lambda x, \lambda y) \equiv \lambda w(x, y)$. The freedom to redefine the dimensionful constants, μ_r has been used to absorb constants which could have appeared additively in each of the square brackets.

The constants a_r and b_r are determined by requiring $U_r^{a_r} M^{b_r}/L_r^2$ to be invariant with respect to the abelian global symmetries. For an $SU(N_1) \times SU(N_2)$ model with one generation of nonchiral matter in the fundamental representation of both gauge-group factors this implies:

$$a_1 = -a_2 = N_1 - N_2\,, \qquad b_1 = N_2 \qquad \text{and} \qquad b_2 = N_1\,. \tag{2.12}$$

Finally, the otherwise undetermined function, $w(x, y)$, may be fixed by requiring W to reduce to the result for two decoupled pure gauge theories, with no matter:

$$W_{\rm dec} = \frac{1}{4}\Big(U_1 S_1 + U_2 S_2\Big) + \frac{1}{32\pi^2}\left[N_1 U_1 \log\left(\frac{U_1}{\tilde\mu_1^3}\right) + N_2 U_2 \log\left(\frac{U_2}{\tilde\mu_2^3}\right)\right], \tag{2.13}$$

in the decoupling limit, where the quark mass, m, goes to infinity. In this way one finds the result: $w(x, y) = -\frac{1}{32\pi^2}\Big[N_2\, x\, \log(N_2 + N_1 y/x) + N_1\, y\, \log(N_1 + N_2 x/y)\Big]$, allowing the full superpotential to be written:

$$\begin{aligned} W = \frac{1}{4}\Big(U_1 S_1 + U_2 S_2\Big) + mM &+ \frac{U_1}{32\pi^2}\left[N_1 \log\left(\frac{U_1}{\mu_1^3}\right) - N_2 \log\left(\frac{N_2 U_1 + N_1 U_2}{M\mu_1}\right)\right] \\ &+ \frac{U_2}{32\pi^2}\left[N_2 \log\left(\frac{U_2}{\mu_2^3}\right) - N_1 \log\left(\frac{N_2 U_1 + N_1 U_2}{M\mu_2}\right)\right]. \end{aligned} \tag{2.14}$$

It is instructive to write out the condition which is obtained if this expression is extremized with respect to M. So long as $M \neq 0$, the condition $\partial W/\partial M = 0$ implies

$$mM + \frac{1}{32\pi^2}\Big(N_2 U_1 + N_1 U_2\Big) = 0\,, \tag{2.15}$$

which may be recognized as the Konishi anomaly. This anomaly follows automatically from the effective superpotential as a consequence of supersymmetry and our imposition of the anomalous $U(1)$ symmetries.

Using eq. (2.15) to eliminate M gives,[4] *by construction*, the decoupled expression, eq. (2.13), up to an additive constant. The scales $\tilde{\mu}_r$ are given by $\tilde{\mu}_1^3 = \mu_1^3 \cdot \left|32\pi^2 em/\mu_1\right|^{N_2/N_1}$, with $\tilde{\mu}_2$ given by a similar expression with $1 \leftrightarrow 2$. ($e = 2.7...$ here represents the base of the natural logarithms.) Varying with respect to U_r, for the confining phase we therefore quite generally find the extremal value:

$$\overline{U}_r = \left(\frac{\tilde{\mu}_r^3}{e}\right) \; e^{-8\pi^2 S_r/N_r} , \tag{2.16}$$

and so the superpotential for the dilaton has the standard runaway form:

$$W(S) = -\frac{1}{32\pi^2}\left(N_1\, U_1 + N_2\, U_2\right) = -\frac{1}{32\pi^2 e}\left(N_1\, \tilde{\mu}_1^3\, e^{-8\pi^2 k_1\, S/N_1} + N_2\, \tilde{\mu}_2^3\, e^{-8\pi^2 k_2\, S/N_2}\right), \tag{2.17}$$

where eq. (1.2) has been used to express S_r in terms of the dilaton S.

2.6. The Wilson superpotential for the Coulomb phase

We now turn our attention to the Coulomb phase, for which $m = 0$. Due to the presence of massless modes in this phase only the superpotential for the Wilson action is guaranteed to be local and to be a holomorphic function of its arguments. We make some remarks concerning the superpotential for the effective quantum action in the Coulomb phase in the next section.

The choice of variables: Whereas the arguments of the quantum action are ours to choose, those of the Wilson action must describe the model's low-energy degrees of freedom. As such they can differ for differing phases, even within a given model.

In this section we start by assuming the relevant degrees of freedom to be similar to those which describe the model's low-energy sector for $N_1 > N_2$, in the limit where the $SU(N_2)$ gauge coupling, g_2, is taken to zero. In this limit we have SQCD with $N_f = N_2 < N_c = N_1$, with the low-energy physics described by the N_2^2 meson-like variables, ${\mathcal{M}_\alpha}^\beta = Q_{a\alpha}\widetilde{Q}^{a\beta}$. For nonzero g_2 we must restrict these to be $SU(N_2)$ invariant, and so restrict our attention to the eigenvalues, λ_p, $p = 1, \ldots, N_2$, of the matrix ${\mathcal{M}_\alpha}^\beta$.

Notice that, for $N_1 \geq N_2 + 2$, these N_2 eigenvalues are precisely what is required to parameterize the model's semiclassical $D^\flat$ directions even when g_2 is nonzero. For the cases $N_1 = N_2$ or $N_1 = N_2 + 1$, there are $N_2 + 1$ $D^\flat$ directions, and so another

[4] Since W is the superpotential for the quantum action — as opposed to the Wilson action — the correct procedure for 'integrating out' fields is to remove them by solving their extremal equations, rather than by performing their path integral. Furthermore, for supersymmetric theories in the low-energy limit when the fields being eliminated do not acquire supersymmetry-breaking v.e.v.'s, this should be done using the effective superpotential, W, rather than the effective scalar potential V [21].

invariant is required. For example when $N_1 = N_2$ we may choose this to be the baryonic invariants

$$B \equiv Q_{a_1\alpha_1} \cdots Q_{a_{N_2}\alpha_{N_2}} \; \epsilon^{a_1\cdots a_{N_2}} \; \epsilon^{\alpha_1\cdots\alpha_{N_2}} \,, \quad \widetilde{B} \equiv \widetilde{Q}^{a_1\alpha_1} \cdots \widetilde{Q}^{a_{N_2}\alpha_{N_2}} \; \epsilon_{a_1\cdots a_{N_2}} \; \epsilon_{\alpha_1\cdots\alpha_{N_2}} \,. \tag{2.18}$$

These last two quantities are not both independent, since classically $B\widetilde{B}$ may be expressed as a function of the λ_p's.

For future use, we note in passing that one might choose a different way to express the invariants λ_p. This is by using quantities: $M_p \equiv \mathrm{Tr}\,(\mathcal{M}^p) = \sum_k \lambda_k^p$, for $p = 1, \ldots, N_2$. As we shall see, there are situations for which these variables have different physical implications, and so care must be used when interchanging the variables λ_p for M_p. In what follows, whenever we must choose we use the eigenvalues λ_p, rather than the variables M_p.

Symmetry constraints for the Wilson superpotential: Finally, notice that the dependence of the Wilson superpotential on the invariants, λ_p, is greatly simplified by considering its behaviour in the limit when $s_2 \to \infty$, because of the enhanced flavour symmetry which emerges in this limit.

Consider first the case $N_1 \geq N_2 + 2$. In this case invariance of the Wilson action with respect to the global $U(1)$ symmetries implies the result must have the form:

$$W(S_1, S_2, \lambda_p) = \left(\frac{L_1^2}{\lambda_1 \cdots \lambda_{N_2}} \right)^{1/(N_1 - N_2)} \Omega(z_1, \ldots, z_p) \,, \qquad \text{if } N_1 \geq N_2 + 2 \,, \tag{2.19}$$

where Ω is an arbitrary function of the invariants $z_p \propto L_1 L_2 / \lambda_p^{(N_1+N_2)/2}$.

Now if $L_2 \to 0$ with $L_1 L_2$ fixed, then the promotion of the gauge $SU(N_2)$ symmetry to the global flavour group $SU(N_2) \times \widetilde{SU}(N_2)$ implies the unknown function Ω can depend on the λ_p only through the combination $\det \mathcal{M}$. (For example, for $N_2 = 2$, Ω can depend on λ_1 and λ_2 [or: M_1 and M_2] only through the combination $\det \mathcal{M} = \lambda_1 \lambda_2$ [or: $\det \mathcal{M} = \frac{1}{2}\,(M_1^2 - M_2)$].) It follows that agreement with this limit implies eq. (2.19) can be sharpened to involve an unknown function of only one variable:

$$W(S_1, S_2, \lambda_p) = \left(\frac{L_1^2}{\det \mathcal{M}} \right)^{1/(N_1 - N_2)} \Omega(z), \qquad \text{if } N_1 \geq N_2 + 2 \,. \tag{2.20}$$

where now Ω depends only on $z \propto L_1 L_2 / (\det \mathcal{M})^{(N_1+N_2)/2N_2}$.

The symmetry consequences for the Wilson superpotential are even more striking for the case where $N_1 = N_2$. In this case the same arguments as those just given imply that there is no superpotential at all which is consistent with all of the symmetries of the problem. This is because the quantum numbers for the fields in this case ensure that any quantity which is $U_A(1)$ invariant must also be $U_R(1)$ invariant. But this is inconsistent for the superpotential, which must be invariant with respect to $U_A(1)$, but carry charge 2 with respect to $U_R(1)$. We conclude:

$$W(S_1, S_2, \lambda_p, B, \widetilde{B}) = 0 \,, \qquad \text{if } N_1 = N_2 \,. \tag{2.21}$$

The limits $g_r \to 0$: More information about the function $\Omega(z)$ is obtained by examining the limits when either of the gauge couplings is set to zero. Consider first the limit where g_2, and so also L_2, vanishes, for simplicity restricting our attention to the case $N_1 \geq N_2 + 2$. With these choices W must approach the appropriate limit, $W \propto (L_1^2/\det \mathcal{M})^{1/(N_1-N_2)}$, for SQCD with $N_f = N_2$ and $N_c = N_1$ flavours. Agreement with this limit clearly implies $\Omega(z) \to$ constant as $z \to 0$.

Notice, however, that this small-z behaviour for $\Omega(z)$ implies that W *cannot* approach a similar finite limit which depends only on $\det \mathcal{M}$ and L_2 as $L_1 \to 0$. The unique such result consistent with the flavour symmetries is $W \propto \left(L_2^2/(\det \mathcal{M})^{N_1/N_2}\right)^{1/(N_2-N_1)}$, which cannot be obtained if $\Omega \sim$ constant for small z. The absence of such a limit as $L_1 \to 0$ is just as well, however, because the microscopic theory in this limit is SQCD with $N_f \geq N_c + 2$ generations, whose low-energy limit is known not to be well-described simply by variables $\propto Q\widetilde{Q}$ [1]. We should therefore expect a transition to another phase to qualitatively change the low-energy spectrum when L_1 is sufficiently small.

2.7. The quantum action for the Coulomb phase

In order to better understand the Wilson superpotential in the Coulomb phase it is worth imagining it to have been obtained from a quantum effective action by extremizing with respect to the gaugino fields, U_r.[5] Although this procedure might seem suspect, given the possibility for nonlocal contributions and holomorphy anomalies, these complications have been argued in ref. [22] to be irrelevant under certain circumstances.

Consider, then, the form an effective W must take consistent with (*i*) the model's global symmetries; (*ii*) the limiting behaviour for $S_r \to \infty$; and (*iii*) its linear dependence on S_r. Repeating the steps taken when analysing the confining phase gives in this case the following result for W:

$$\begin{aligned} W(S_r, U_r, \lambda_p) = {} & \frac{1}{4}\Big(U_1 S_1 + U_2 S_2\Big) + \frac{U_1}{32\pi^2}\left[N_1 \log\left(\frac{U_1}{\mu_1^3}\right) - N_2 \log\left(\frac{N_2 U_1 + N_1 U_2}{(\det \mathcal{M})^{1/N_2}\mu_1}\right)\right] \\ & + \frac{U_2}{32\pi^2}\left[N_2 \log\left(\frac{U_2}{\mu_2^3}\right) - N_1 \log\left(\frac{N_2 U_1 + N_1 U_2}{(\det \mathcal{M})^{1/N_2}\mu_2}\right)\right], \end{aligned} \tag{2.22}$$

where $\det \mathcal{M} = \prod_p \lambda_p$.

An interesting feature of eq. (2.22) is that it is *not* equally valid to regard it as depending on the two sets of variables λ_p and M_p. This may be seen by adding a quark mass term — either mM_1 or $m\,(\lambda_1 + \cdots + \lambda_{N_2})$ — and then eliminating these variables to obtain the superpotential for S_r and U_r only. For nonzero m this must reproduce the decoupled form of eq. (2.13). Although eq. (2.13) *is* reproduced if the λ_p are used as

[5] This procedure is called 'integrating in' in the second reference of ref. [1].

independent variables, it is *not* obtained using the M_p. In fact, there are no solutions at all to $\partial W/\partial M_p = 0$, because the mass term depends only on M_1, whereas the nonperturbative superpotential depends on all the M_p's only through the combination $\det \mathcal{M}$. (This situation is not improved by adding higher order perturbative terms to W.) Uncritical use of the variables M_p, would lead us to conclude mistakenly that supersymmetry is spontaneous broken. It is possible to obtain different physical results like this, simply by using two different variables, because the change of variables from λ_p to M_p is not linear and it happens that the Jacobian, $\partial(\lambda_1, \cdots, \lambda_{N_2})/\partial(M_1, \cdots, M_{N_2})$, vanishes at the solution to the stationary condition $\partial W/\partial \lambda_p = 0$.

If we denote by $\overline{U}_r$ the stationary points of eq. (2.22) with respect to variations of the U_r, then:

$$\begin{aligned} \frac{\overline{U}_1^{\,N_1}}{(N_2\overline{U}_1 + N_1\overline{U}_2)^{N_2}} &= \frac{\kappa_1 L_1^2}{\det \mathcal{M}} \\ \frac{\overline{U}_2^{\,N_2}}{(N_2\overline{U}_1 + N_1\overline{U}_2)^{N_1}} &= \frac{\kappa_2 L_2^2}{(\det \mathcal{M})^{N_1/N_2}}\,, \end{aligned} \tag{2.23}$$

with the constants $\kappa_{1,2}$ defined as $\kappa_1 \equiv \mu_1^{3N_1-N_2}\, e^{N_1-N_2}$, $\kappa_2 \equiv \mu_2^{3N_2-N_1}\, e^{N_2-N_1}$.

Using these expressions the superpotential can be written as:

$$W(S_1, S_2, \lambda_p) = W(\overline{U}_1, \overline{U}_2, \det \mathcal{M}, S_1, S_2) = -\,\frac{(N_1 - N_2)}{32\pi^2}\left(\overline{U}_1 - \overline{U}_2\right), \tag{2.24}$$

so the explicit solution of $\overline{U}_1$ and $\overline{U}_2$, using (2.23), gives the desired superpotential as a function of L_1, L_2 and M. Notice how the result would vanish (as expected) if N_1 were to equal N_2.

It only remains to solve for $\overline{U}_1$ and $\overline{U}_2$. To do so we start by incorporating the global $U(1)$ symmetries:

$$\overline{U}_1 = \left(\frac{\kappa_1 L_1^2}{\det \mathcal{M}}\right)^{1/(N_1-N_2)} f_1(z)\,, \qquad \overline{U}_2 = \left(\frac{\kappa_2 L_2^2}{(\det \mathcal{M})^{N_1/N_2}}\right)^{1/(N_2-N_1)} f_2(z)\,, \tag{2.25}$$

where we sharpen our earlier definition, and write $z = \sqrt{\kappa_1\kappa_2}\; L_1 L_2/(\det \mathcal{M})^{(N_1+N_2)/2N_2}$. Moreover, eqs. (2.23) imply $f_2(z) = z^{2N_1/N_2(N_1-N_2)}\,[f_1(z)]^{N_1^2/N_2^2}$, so it suffices to solve for $f_1(z)$. This function is determined by eq. (2.23) as the solution to the following algebraic equation:

$$X^{N_1^2/N_2^2} - 3\lambda\, z^{-2/(N_1+N_2)}\, X^{N_1/N_2} + N_2\, X = 0\,, \tag{2.26}$$

where

$$X(z) = N_1^{N_2^2/(N_1^2-N_2^2)}\, z^{2N_2/(N_1^2-N_2^2)}\, f_1(z) \qquad \text{and} \qquad 3\,\lambda = N_1^{-N_2/(N_1+N_2)}\,. \tag{2.27}$$

Eq. (2.26) cannot be solved in closed form for arbitrary values of N_1 and N_2, which precludes the explicit evaluation of eqs. (2.24) in the general case. (By contrast, it is an interesting advantage of the U_r-dependent superpotential, eq. (2.22), that it *can* be found explicitly.) We therefore defer further perusal of the solution to the following sections, where we focus on simple special cases. In particular, section 4 examines in detail the case $N_1 = 4$ and $N_2 = 2$, for which eq. (2.26) is cubic, and so may be explicitly solved.

3. The $SU(2) \times SU(2)$ model

Let us now specialize to the simple case $N_1 = N_2 = 2$, which is also examined in ref. [16]. (Our results in this section essentially reproduce those of this reference.) In this case because the representation $(\mathbf{2}, \mathbf{2})$ is pseudoreal, we regard our matter content to be $Q_{ia\alpha} \in (\mathbf{2}, \mathbf{2})$, where $i = 1, 2$ is a flavour index. The classical flavour symmetry of the microscopic theory in the absence of quark masses is therefore $G_f = SU_f(2) \times U_A(1) \times U_R(1)$ (with $U_B(1) \subset SU_f(2)$). Anomalies break the two $U(1)$ symmetries down to the anomaly-free R-symmetry whose charge for superfields is $\widetilde{R} = R - \frac{2}{3}\, A$, and so for which $\widetilde{R}(Q_{ia\alpha}) = 0$.

In this case there are $N_2 + 1 = 3$ $D^\flat$ directions, which we may parameterize using the symmetric matrix, $M_{ij} = Q_{ia\alpha} Q_{jb\beta}\, \epsilon^{ab}\, \epsilon^{\alpha\beta}$. The flavour symmetries imply the Wilson superpotential only depends on these variables through the single combination $\det M$. There are two independent invariant quantities with respect to the two global $U(1)$ symmetries, which we take to be L_1/L_2 and $\xi \propto \det M/(L_1\, L_2)$.

3.1. The confining phase

For nonzero quark masses, m, we expect a confining phase and so compute the superpotential for the quantum effective action. The unique such superpotential consistent with the symmetries, linearity and which gives a decoupled result for large m is:

$$
\begin{aligned}
W = {} & \frac{U_1}{32\pi^2} \left[\log\left(\frac{\det M}{\Lambda_1^4} \right) - 2 \log\left(\frac{U_1}{U_1 + U_2} \right) \right] \\
& + \frac{U_2}{32\pi^2} \left[\log\left(\frac{\det M}{\Lambda_2^4} \right) - 2 \log\left(\frac{U_2}{U_1 + U_2} \right) \right] + \mathrm{Tr}\left(mM \right),
\end{aligned}
\tag{3.1}
$$

where $\Lambda_r^2 \equiv \mu^2 \, L_2/\left(32\pi^2 e^{\frac{1}{2}}\right)$, defines the RG-invariant scale for each gauge-group factor.

If eq. (3.1) is first extremized with respect to M_{ij}, we find the stationary point to be $(M^{-1})_{ij} = -32\pi^2\, m_{ij}/(U_1 + U_2)$. When this is substituted back into W we obtain the usual decoupled result, eq. (2.13), with U_r given by $U_r \propto \Lambda_r^2\, \sqrt{\det m}$. We obtain in this way a runaway dilaton potential, as expected.

Different information may be extracted from eq. (3.1) if the U_r are instead eliminated before M_{ij}. The saddle point conditions $\partial W/\partial U_r = 0$ imply $U_1/U_2 = \pm\, L_1/L_2 = \pm\, \Lambda_1^2/\Lambda_2^2$, together with the 'quantum constraint':

$$
\det M = \left(\Lambda_1^2 \pm \Lambda_2^2 \right)^2 . \tag{3.2}
$$

In the limit in which either L_1 or L_2 vanish, this constraint reduces to the well-known quantum constraint of SQCD when $N_c = N_f$, and it is precisely what is required to ensure the matching of the B and $\widetilde{R}$ anomalies in the confining phase for vacua having $M_{11} = M_{22} = 0$, $M_{12} \neq 0$.

3.2. The Coulomb phase

Semiclassically, in the absence of quark masses the three $D^\flat$ directions do not get lifted, along which the gauge group $SU(2) \times SU(2)$ is broken to an unbroken, diagonal $U(1)$. Furthermore, constraint (3.2) does not apply to this phase, so the massless degrees of freedom one infers for the model therefore comprises one $U(1)$ gauge supermultiplet plus the three gauge-neutral matter multiplets contained in M_{ij}. As is easily verified, the additional gauge multiplet cancels the contributions of M_{12} to the B and $\widetilde{R}$ anomalies, thereby ensuring these anomalies continue to match in the Coulomb phase even though constraint (3.2) no longer applies there.

We now construct the Wilson action's superpotential and gauge coupling function for these degrees of freedom. Although our results here reproduce those of ref. [16], we spell them out to facilitate our presentation of the $SU(4) \times SU(2)$ model of the next section.

The superpotential: As described in section 2 above, because of the absence of the quark mass matrix the Wilson superpotential for the M_{ij} and the dilaton is forced to vanish by the model's $U(1)$ flavour symmetries:

$$W(L_r, M_{ij}) = 0\,. \tag{3.3}$$

The gauge coupling function: We now turn to the coupling function, $S_{\rm eff} \equiv -\frac{i}{4\pi}\,\tau(S_1, S_2, M_{ij})$, for the low-energy $U(1)$ gauge multiplet. Here we normalize τ so that its relationship with the effective $U(1)$ gauge coupling and Θ-angle is $\tau = \frac{\Theta_{\rm eff}}{2\pi} + \frac{4\pi i}{g^2_{\rm eff}}$. Our construction follows that of ref. [23].

To construct τ, we look for a function having the following properties:

- *Positivity:* Because τ appears in the gauge kinetic terms for the $U(1)$ gauge mode of the low-energy effective action, its imaginary part must be positive.

- *Duality:* τ transforms in the standard way under the duality transformations of the low energy effective theory:

$$\tau \to \frac{A\tau + B}{C\tau + D}\,, \tag{3.4}$$

  together, possibly, with an action on the other moduli, such as M_{ij}. If A, B, C and D are arbitrary integers, then the duality group is $PSL(2, Z)$. Otherwise it is a subgroup of this group.

- *Global symmetries:* Invariance of the Wilson action with respect to the global flavour symmetries of the microscopic model is ensured if τ depends on L_r and M_{ij} only through the invariant combinations L_1/L_2 and $\xi \equiv \det M/\Lambda_1^2\,\Lambda_2^2$.

- *Singularities:* Like other terms in the Wilson action, τ may develop singularities at points in the moduli space where otherwise massive states come down and become massless. Here we make the key assumption that the singularities of τ are: (i) at weak coupling ($\xi \to \infty$); and (ii) at the confinement points — $\xi = \xi_\pm \equiv (\Lambda_1^2 \pm \Lambda_2^2)^2/\Lambda_1^2\,\Lambda_2^2$ — and nowhere else. (Notice $\xi_\pm$ satisfy the identity $\xi_+ - \xi_- \equiv 4$.) The singularities at these points are argued in ref. [16] to be due to the masslessness there of various monopole degrees of freedom whose condensation is responsible for the onset of confinement.

  For later purposes we remark that the gaugino condensates, $\overline{U}_r$, are also not analytic at the singular points $\xi = \xi_\pm$, since they are nonzero in the confining phase, but vanish whenever $\xi \neq \xi_\pm$.

- *The weak-coupling limit:* We require, at weak coupling, that the effective coupling, $S_{\text{eff}} = -\,i\tau/4\pi$ approach the bare coupling, $S_1 + S_2$, corresponding to the unbroken $U(1)$ of the gauge group.

- *Nonsingular β function:* Finally, we require the β function, $\beta(\tau) = (\mu^2\partial\tau/\partial\mu^2)_{M,L_r}$ to have no poles for $\Im\tau \neq 0$.

These assumptions (which need not all be independent) are satisfied by the geometrical solution for τ which was introduced in ref. [23]. This solution is obtained by taking τ to be the modulus of the torus which is defined by a cubic curve,

$$y^2 = x^3 + a\,x^2 + b\,x + c\,, \tag{3.5}$$

in the two-dimensional complex plane with parameters a, b, and c given as functions of the moduli ξ and L_1/L_2. These functions are chosen to ensure that this torus is singular only at the assumed points: $\xi \to \infty$ and $\xi = \xi_\pm$.

Keeping in mind the identity $\xi_+ - \xi_- = 4$, a choice which satisfies all of the above conditions is:

$$a = \frac{1}{2}\Big(\xi_+ + \xi_-\Big) - \xi = \xi_- + 2 - \xi\,, \qquad b = \frac{1}{4}\Big(\xi_+ - \xi_-\Big) = 1\,, \qquad c = 0\,. \tag{3.6}$$

The singularities of the resulting torus are determined by the vanishing (or divergence) of the discriminant:

$$\begin{aligned} \Delta(a,b,c) &\equiv 4b^3 - a^2\,b^2 - 18abc + 4a^3\,c + 27c^3 \\ &= (\xi - \xi_-)\,(\xi_- + 4 - \xi)\,. \end{aligned} \tag{3.7}$$

Given such an elliptic curve, the gauge coupling function may be implicitly constructed using the standard modular-invariant function $j(\tau)$, through the relation:

$$j(n\tau) = \frac{6912(b - \frac{1}{3}\,a^2)^3}{\Delta(a,b,c)} = 256\,\frac{[(\xi - \xi_- - 2)^2 - 3]^3}{(\xi - \xi_-)\,(\xi - \xi_- - 4)}\,. \tag{3.8}$$

Here n is to be chosen to ensure that $S_{\rm eff} \to S_1 + S_2$ in the weak-coupling (*i.e.* large-ξ) limit.

This choice determines the model's exact β-function, defined by the variation of τ with μ as the moduli M_{ij} and L_r are held fixed, in a similar way as has been found for $N = 2$ supersymmetric $SU(2)$ gauge theory [25]. Taking $(\mu^2\partial/\partial\mu^2)_{M,L_r}$ of both sides of eq. (3.8), and using $\mu^2\partial\xi/\partial\mu^2 = -2\,\xi$, gives:

$$\beta(\tau) \equiv \mu^2\left(\frac{\partial\tau}{\partial\mu^2}\right)_{M,L_r} = 512\,\frac{(\xi - \xi_- - 2)\,[(\xi - \xi_- - 2)^2 - 3]^2\,[2(\xi - \xi_- - 2)^2 - 9]}{(\xi - \xi_-)^2\,(\xi - \xi_- - 4)^2\;n\,j'(n\tau)}\,. \tag{3.9}$$

We next list several of the properties of $j(\tau)$ and $j'(\tau)$, which are useful for extracting the properties of the functions $\tau(\xi)$ and $\beta(\tau)$. It suffices to specify these within a fundamental domain, F, obtained by identifying points in the upper-half τ-plane under the action of $PSL(2,Z)$. (We choose for F the interior, and part of the boundary, of the standard strip, defined by $\Im\tau > 0$, $|\Re\tau| < \frac{1}{2}$ and $|\tau| > 1$.) In fact, $j(\tau)$ furnishes a one-to-one map from F to the complex Reimann sphere [24]. The properties of interest are:

- *Poles and zeroes:* The positions of all of the poles and zeroes of $j(\tau)$ and $j'(\tau)$ are known. Neither $j(\tau)$ nor $j'(\tau)$ have any singularities for finite τ within F. $j(\tau)$ has a triple zero at the edges of F, when $\tau = e^{i\pi/3}$ (plus $PSL(2,Z)$ transformations of this point), and so $j'(\tau)$ also has a double zero here. $j'(\tau)$ has an additional zero at $\tau = i$, since near this point $j(\tau) = 1728 + O[(\tau - i)^2]$.

- *Asymptota:* As $\tau \to i\infty$, $j(\tau)$ has the following behaviour:

$$j(\tau) = \frac{1}{q} + 744 + 196884\,q + O(q^2)\,, \tag{3.10}$$

  where $q \equiv e^{2\pi i\tau}$. It follows that $j'(\tau) = -2\pi i/q + \cdots$ in the same limit.

Using these properties it is straightforward to verify the following properties:

- *Unphysical poles:* Notice first that the zeroes of $j'(\tau)$ in the denominator of eq. (3.9) are cancelled by zeroes of its numerator, leaving a well-behaved expression as $n\tau \to e^{i\pi/3}$ and $n\tau \to i$.

- *Fixing $n = 2$:* To fix n we examine the weak-coupling (large-ξ) limit in more detail. Combining eqs. (3.8) and (3.10) we find $j(n\tau) = 1/q^n + \cdots = 256\,\xi^4 + \cdots$. This is consistent with $S_{\rm eff} = S_1 + S_2 + \cdots$ (and so $q \propto \xi^{-2}$) only if $n = 2$. (Notice this agrees with ref. [16], once their normalization $\tau_{16} = \Theta_{\rm eff}/\pi + 8\pi i/g^2_{\rm eff} = 2\tau$ is taken into account.)

- *The perturbative β-function:* Given the choice $n = 2$, we may read off the weak-coupling limit of the β-function defined by eq. (3.9). Since eq. (3.8) implies $\xi \sim 1/\left(4\, q^{\frac{1}{2}}\right)$ for $\tau \to i\infty$, we see that in the same limit eq. (3.9) becomes:

$$\beta(\tau) = -\,\frac{2i}{\pi} + O(q)\,. \tag{3.11}$$

The constant term in this expression corresponds to a one-loop result. Higher-loop contributions to β would be proportional to powers of $1/\tau$ and so are seen to be zero, in agreement with standard nonrenormalization results. The subleading terms in eq. (3.11) are $O(q)$ and so express instanton contributions to the coupling-constant running.

The one-loop contribution to eq. (3.11) is to be compared with the general one-loop renormalization-group running of the gauge couplings. The standard one-loop expression is:

$$\frac{4\pi}{g^2(\mu')} = \frac{4\pi}{g^2(\mu)} + \frac{1}{12\pi}\left[11\,T(A) - 2\,T(\mathcal{R}_{\frac{1}{2}}) - T(\mathcal{R}_0)\right]\,\log\left(\frac{\mu'^2}{\mu^2}\right), \tag{3.12}$$

where $\mathcal{R}_{\frac{1}{2}}$ denotes the gauge representation for the model's left-handed spin-half particles and $\mathcal{R}_0$ is the gauge representation for its complex scalar fields. (As before A represents the adjoint representation, as is appropriate for the spin-one particles.) Since the scale μ of eq. (3.11) represents a scale in the macroscopic, low-energy theory, it plays the role taken by μ (rather than μ') in eq. (3.12). Specializing now to the microscopic $SU(N_1) \times SU(N_2)$ supersymmetric gauge theory gives:

$$\mu^2\frac{\partial}{\partial\mu^2}\left(\frac{4\pi}{g_1^2}\right) = \frac{N_2 - 3N_1}{4\pi}\,, \qquad \mu^2\frac{\partial}{\partial\mu^2}\left(\frac{4\pi}{g_2^2}\right) = \frac{N_1 - 3N_2}{4\pi}\,. \tag{3.13}$$

Adding eqs. (3.13) to one another, gives the one-loop contribution:

$$\beta(\tau) = -\,\frac{i}{2\pi}\,(N_1 + N_2)\,, \tag{3.14}$$

which clearly agrees with eq. (3.11) once specialized to the special case $N_1 = N_2 = 2$.

3.3. Dilaton dependence

The low-energy dilaton dynamics is inferred by now re-expressing S_1 and S_2 in terms of S. If $k_1 = k_2 \equiv k$, as is usually the case, this implies $L_1 = L_2 = L \equiv \exp\left[-4\pi^2 kS\right]$. With this choice we see that the confining points of the space of moduli become $\xi_- = 0$ and $\xi_+ = 4$.

The vanishing of the Wilson superpotential in the Coulomb phase for this (and any other $N_1 = N_2$) model makes the low-energy dilaton dynamics particularly simple. All that remains is to perform the path integration over the remaining massless degrees of freedom. On performing these integrations, for any globally supersymmetric model the flat directions of the Wilson superpotential also become flat directions of the exact quantum superpotential, so long as the dynamics of the massless modes does not spontaneously break supersymmetry. Since such supersymmetry breaking is forbidden to all orders in perturbation theory, and is protected nonperturbatively by supersymmetric index theorems [26], it is extremely unlikely in this model. We therefore expect the exact superpotential for S to remain precisely flat. Unfortunately, less may be said about the shape of the dilaton superpotential away from the flat directions. This is because this shape can depend on the (unknown) Kähler potential of the low-energy theory, or on its (calculable) gauge coupling function.

4. The $SU(4) \times SU(2)$ model

Let us now consider the case $N_1 = 4$, $N_2 = 2$, which is the simplest example of the class of models for which $N_1 \geq N_2 + 2$. In this case there are $N_2 = 2$ $D^\flat$ directions, which we may parameterize using the invariant eigenvalues, λ_1 and λ_2, of the two-by-two matrix, $\mathcal{M}_\alpha{}^\beta = Q_{a\alpha}\widetilde{Q}^{a\beta}$. As discussed in previous sections, the Wilson superpotential only depends on these two variables through a single combination, which we denote by $M = (\lambda_1\lambda_2)^{\frac{1}{2}}$.

The analysis of the confining phase for this model proceeds just as for the general case, as described in section 2. We therefore focus here on the Coulomb phase of the model.

The invariant quantity with respect to the global $U(1)$ symmetries in this case is $z = \sqrt{\kappa_1\kappa_2}\; L_1\, L_2/M^3$, and expressions (2.25) and (2.27) become:

$$\begin{aligned} \overline{U}_1 &= \left(\frac{\sqrt{\kappa}_1\; L_1}{M}\right) f_1(z) = \left(\frac{\kappa_1 L_1^2}{4\sqrt{\kappa_2}\; L_2}\right)^{1/3} X(z)\,, \\ \overline{U}_2 &= \left(\frac{M^2}{\sqrt{\kappa_2} L_2}\right) f_2(z) = \frac{1}{4}\left(\frac{\kappa_1 L_1^2}{4\sqrt{\kappa_2}\; L_2}\right)^{1/3} X^4(z)\,. \end{aligned} \tag{4.1}$$

The anomaly-free discrete discrete symmetry, eq. (2.10), is in this case the group $Z_4 \times Z_4 \subset U_A(1) \times U_R(1)$, where:

$$\frac{\beta_A}{2\pi} = \frac{k_A}{4} \qquad \text{and} \qquad \frac{\beta_R}{2\pi} = \frac{3k_R}{4}\,, \tag{4.2}$$

where k_A and k_R are integers. The action of this discrete symmetry on the fields U_r, L_r and M therefore is:

$$U_r \to e^{i\pi k_R}\, U_r\,, \qquad \lambda_p \to e^{i\pi k_A}\,\lambda_p\,, \qquad L_1 \to e^{i\pi(k_A+5k_R)}\, L_1\,, \qquad L_2 \to e^{i\pi(2k_A+k_R)}\, L_2\,. \tag{4.3}$$

Clearly vacua for which U_r and λ_p are nonzero must come in 4-dimensional $Z_4 \times Z_4$ multiplets which differ by an overall sign change for the λ_p and the U_r.

4.1. Monodromy

For this example the superpotential can be explicitly found, since the algebraic equation, eq (2.26), determining X either implies $X = 0$ or X is the solution to the cubic:

$$X^3 - 3\,\xi\,X \;+\; 2 \;=\; 0\,, \tag{4.4}$$

where $\xi \;=\; \lambda\,z^{-1/3}$, and $3\lambda \;=\; 4^{-1/3}$. The explicit solutions of this equation can be written in terms of the quantities

$$T_\pm^3 \;=\; -1 \pm \sqrt{1-\xi^3}\,, \tag{4.5}$$

with the three solutions given by:

$$X_n \equiv X(\xi;\rho_n) = \rho_n\,T_+ \;+ \rho_n^2\,T_-\,, \tag{4.6}$$

with ρ_n being the three roots of unity, $\rho_n = e^{2\pi i n/3}$, $n = 0,1,2$.

We next establish that the three roots of this cubic equation can be related to one another by simultaneously shifting the two vacuum angles, Θ_r, through 4π radians (which is *not* a $Z_4 \times Z_4$ transformation). We do so by showing that such a shift can be interpreted as a monodromy transformation about the branch point the solutions have at $\xi^3 = 0$, or $\xi^3 = \infty$.

Consider therefore performing the simultaneous shift $\Theta_r \;\to\; \Theta_r + 4\pi$ for both of the vacuum angles. Under such a shift the argument of L_r shifts by 2π, but $\xi \propto (L_1 L_2)^{-1/3}$ acquires the phase: $\xi \;\to\; \omega\,\xi$, where $\omega \;=\; e^{2\pi i/3}$. This is an invariance of the equation defining X_n, eq. (4.4), provided that X_n also transforms by $X_n \to \omega^2 X_{n'}$, since this ensures ω cancels in both X^3 and ξX. (Inspection of eq. (4.1) shows it also leaves U_r unchanged because the transformation $(L_1^2/L_2)^{1/3} \to \omega\,(L_1^2/L_2)^{1/3}$ cancels the transformation of X.)

This shift may be regarded as a monodromy transformation since it takes $\xi^3 \to e^{2\pi i}\;\xi^3$, thereby circling the branch point in the complex ξ^3 plane once. We now compute its action on X_n. From eq. (4.5) we see $T_\pm^3\,(e^{2\pi i}\,\xi^3) = T_\mp^3(\xi)$, so if we define the branch of the cube root so that $T_\pm\,(e^{2\pi i}\,\xi^3) = \omega^2\;T_\mp(\xi)$ then we find the monodromy transformation:

$$X(\omega\,\xi;\rho_n) = \omega^2\;X(\xi;\rho_n^2)\,. \tag{4.7}$$

4.2. Extremizing with respect to M

Substituting any of these solutions back into the superpotential gives, once we use equations (2.24), (2.25) and (4.4):

$$W_n(S_1,S_2,\lambda_1,\lambda_2) \;=\; -\,\frac{1}{64\pi^2}\left(\frac{\kappa_1 L_1^2}{4\sqrt{\kappa_2}L_2}\right)^{1/3}\;\left(4 - X_n^3\right)\;X_n\,, \tag{4.8}$$

for $n = 0,1,2$.

Eq. (4.8) defines the superpotential as a function of $M = (\lambda_1\lambda_2)^{\frac{1}{2}}$. Although λ_1 and λ_2 cannot be separately determined, the extremal condition for M is:

$$0 = \frac{\partial W_n}{\partial M} = \left(\frac{\partial W_n}{\partial X_n}\right)\left(\frac{\partial X_n}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial M}\right), \tag{4.9}$$

where

$$\begin{aligned} \left(\frac{\partial W_n}{\partial X_n}\right) &= -\frac{1}{16\pi^2}\left(\frac{\kappa_1 L_1^2}{4\sqrt{\kappa_2}L_2}\right)^{1/3}\left(1 - X_n^3\right), \\ \left(\frac{\partial X_n}{\partial \xi}\right) &= \frac{3X_n^2}{2(X_n^3-1)}, \\ \left(\frac{\partial \xi}{\partial M}\right) &= \frac{\lambda}{(\sqrt{\kappa_1\kappa_2}L_1L_2)^{1/3}}. \end{aligned} \tag{4.10}$$

These conditions do not vanish for finite ξ and X_n, so we are led to examine the asymptotic behaviour for large ξ. This limit also permits us to explore in detail the weak-coupling form for these solutions. We require, then, the large-ξ limit of $T_\pm$, which we write:

$$T_\pm = \omega_\pm\left[\xi^{\frac{1}{2}} \pm \frac{i}{3\xi} + O\left(\xi^{-\frac{5}{2}}\right)\right]. \tag{4.11}$$

Here $\omega_\pm^3 = \pm i$. As may be seen from the previous section, or by direct evaluation, $X_n = \rho_n T_+ + \rho_n^2 T_-$ solves eq. (4.4) only if the phases $\omega_\pm$ are related to one another by $\omega_+ = i\omega_-^2$ and $\omega_- = i\sigma$, where σ is an arbitrary cube root of unity. (Our phase convention for taking the cube root of $T_\pm^3$ in the previous section corresponds to the choice $\sigma = 1$.)

With these phase choices in mind we obtain the following large-ξ form for $\overline{U}_1$:

$$\overline{U}_1 = \frac{1}{2\sqrt{3}}(\omega_+\rho_n + \omega_-\rho_n^2)\left[\frac{\sqrt{\kappa_1}ML_1}{\sqrt{\kappa_2}L_2}\right]^{1/2} - i(-\omega_+\rho_n + \omega_-\rho_n^2)\left[\frac{\sqrt{\kappa_1}L_1}{M}\right] + \cdots,;, \tag{4.12}$$

while $\overline{U}_2 = \left[(\kappa_2 L_2^2)^{\frac{1}{2}}/(\kappa_1 L_1^2)\right]\overline{U}_1{}^4$.

The limits $g_r \to 0$ revisited: We may now see explicitly how the model 'chooses' to take a simple form in the limit $L_2 \to 0$, but not to do so for $L_1 \to 0$. That is, one expects in the limit $L_2 \to 0$ that $\overline{U}_2 = 0$ and $\overline{U}_1$ approaches a finite limit which is a function only of M and L_1. Naively, one might also expect a similar situation also as $L_1 \to 0$, where $\overline{U}_1 = 0$ and $\overline{U}_2$ goes to a finite limit depending only on L_2 and M. The key observation is that, although each of the solutions we have obtained for $\overline{U}_r$ indeed satisfies *one* of these limits, *there is no one vacuum which simultaneously satisfies both limits!*

Before exploring its implications, we first establish the validity of this last claim. To do so notice that for one of the three possible choices for $\omega_\pm$ — *i.e.* for $\sigma = \rho_n^2$ — we have $\omega_+\rho_n + \omega_-\rho_n^2 = 0$. With this choice the weak-coupling limit of $\overline{U}_1$ is determined by the subdominant term of eq. (4.12), rather than by the leading term. Depending on this choice, we therefore find the following two possible weak-coupling forms for $\overline{U}_1$ and $\overline{U}_2$:

$$\begin{aligned} \overline{U}_1 &= \frac{1}{2\sqrt{3}}(\omega_+\rho_n + \omega_-\rho_n^2)\left[\frac{\sqrt{\kappa_1}ML_1}{\sqrt{\kappa_2}L_2}\right]^{1/2} + \cdots \\ \overline{U}_2 &= \frac{1}{144}(\omega_+\rho_n + \omega_-\rho_n^2)^4\left[\frac{M^2}{\sqrt{\kappa_2}L_2}\right] + \cdots \qquad \text{if } (\omega_+\rho_n + \omega_-\rho_n^2) \neq 0\,, \end{aligned} \tag{4.13}$$

or

$$\begin{aligned} \overline{U}_1 &= -i(-\omega_+\rho_n + \omega_-\rho_n^2)\left[\frac{\sqrt{\kappa_1}L_1}{M}\right] + \cdots \\ \overline{U}_2 &= (-\omega_+\rho_n + \omega_-\rho_n^2)^4\left[\frac{\kappa_1\sqrt{\kappa_2}L_1^2L_2}{M^4}\right] + \cdots \qquad \text{if } (\omega_+\rho_n + \omega_-\rho_n^2) = 0\,. \end{aligned} \tag{4.14}$$

As advertised, although eq. (4.13) has the expected limiting form as $L_1 \to 0$, it predicts $\overline{U}_1 \to \infty$ in the $L_2 \to 0$ limit. Precisely the opposite is true of eq. (4.14), for which the $L_2 \to 0$ limit is as expected, but where both $\overline{U}_1$ *and* $\overline{U}_2$ vanish as $L_1 \to 0$.

More generally, two branches for X_n have the generic behaviour, eq. (4.13), and so $X \sim \xi^{\frac{1}{2}}$ for large ξ. These two branches are interchanged by the monodromy transformation discussed in the previous section. For the third branch the leading asymptotic behaviour cancels, leaving eq. (4.14) (or $X \sim \xi^{-1}$ for large ξ), and this branch is unchanged by a monodromy transformation.

We can now see how the model chooses eq. (4.14) as its limiting form, thereby ensuring a simple limit $L_2 \to 0$. The model chooses eq. (4.14) once the field M is allowed to relax to minimize its energy. This may be seen by returning to our examination of the superpotential as a function of M. Inspection of eqs. (4.8) and (4.10) shows that the two branches for which $X \sim \xi^{\frac{1}{2}}$ for large ξ do not satisfy $\partial W/\partial M = 0$. The branch having the large-ξ limit $X \sim \xi^{-1}$ *does* satisfy $\partial W/\partial M = 0$ as ξ, and hence also M, tends to infinity, due to the vanishing of X in this limit. Being the sole supersymmetric vacuum of the three, it is therefore the one which is energetically preferred once M is allowed to relax.

As discussed earlier, this choice is what is expected microscopically, since the limiting theory as $L_2 \to 0$ is SQCD with $N_c = 4$ colours and $N_f = 2$ flavours, which is well described by semiclassical $D^\flat$ variables, such as the λ_p we have used. In the other limit, $L_1 \to 0$, however, the microscopic theory is SQCD with $N_c = 2$ and $N_f = 4$. But for $N_c = 2$ and $N_f = 4$ the theory is in the 'conformal window', whose low-energy limit is believed to be controlled by a nontrivial fixed point of the gauge coupling.

4.3. The gauge coupling function

Just like the $SU(2) \times SU(2)$ model of section 3, the $SU(4) \times SU(2)$ model under consideration has a single unbroken $U(1)$ gauge multiplet in the low-energy sector of its Coulomb phase. We now compute the coupling function, $S_{\rm eff} = -i\tau/4\pi$ for this model. Since the logic follows that used in section 3, we describe here only those features which differ from this earlier discussion.

- *Global symmetries:* For the model at hand, the condition of invariance with respect to the global flavour symmetries of the microscopic theory requires τ to depend only on the single invariant quantity defined above: $w \equiv \xi^3 \propto M^3/(L_1 L_2)$.

- *Singularities:* Unlike for the $SU(2)\times SU(2)$ model, in the present case we do not have a quantum constraint which identifies the confining phase as a particular submanifold of the Coulomb-phase moduli. For small m the shallow directions of the scalar potential in the confining phase are described by the same modulus, w, as describes the flat directions of the Coulomb phase.

  For this model we therefore instead identify the singular points of the function $\tau(w)$ by permitting them only where the gaugino condensates, U_r (and hence also the low-energy superpotential, $W(w)$) become singular. Inspection of the explicit solution, eqs. (4.6) and (4.5), shows this to occur when $w = 0$, $w = 1$ or $w \to \infty$.

- *The weak-coupling limit:* Identification of the unbroken $U(1)$ within the microscopic gauge group, $SU(4)\times SU(2)$ again implies the weak-coupling boundary condition: $S_{\rm eff} = S_1 + S_2 + \cdots$.

An elliptic curve which has the required singularities, and which satisfies all of the other requirements of section 3 is given by eq. (3.5), with

$$a = 4\,w - 2\,, \qquad b = 1\,, \qquad c = 0\,, \tag{4.15}$$

for which the discriminant becomes $\Delta = 16\,w\,(1-w)$. The corresponding expression for the gauge coupling function, $\tau(w)$, then is:

$$j(n\tau) = 16\,\frac{[(4w-2)^2-3]^3}{w\,(w-1)}\,. \tag{4.16}$$

Again $n=2$ is required to ensure that $S_{\rm eff}\to S_1+S_2$ in the large-w limit.

Using $\mu^2 \partial w/\partial\mu^2 = -6w$, the corresponding β-function for the model then is:

$$\beta(\tau) = -96\,\frac{(2w-1)\,[(4w-2)^2-3]^2\,[32w\,(w-1)-1]}{w\,(w-1)^2\,n\,j'(n\tau)}\,. \tag{4.17}$$

We remark on the following properties:

- *Unphysical poles:* Eq. (4.17) is well-behaved, with no poles for $\Im\tau > 0$.

- *Fixing $n=2$:* The large-w, large-$\Im\tau$ limit of eq. (3.8) states $j(n\tau)\sim 1/q^n + \cdots = (16\,w)^4 + \cdots$. Consistency with $S_{\rm eff} = S_1 + S_2 + \cdots$ requires $n=2$, and $w \to 1/\left(16\,q^{\frac{1}{2}}\right)$.

- *The perturbative β-function:* With $n = 2$, the weak-coupling limit of eq. (4.17) states:
$$\beta(\tau) = -\,\frac{3i}{\pi} + O(q)\,, \tag{4.18}$$
which agrees with the perturbative nonrenormalization theorems, as well as the one-loop beta functions of eq. (3.13) once this is specialized to the case $N_1 = 4$ and $N_2 = 2$.

4.4. Dilaton dependence

As before, we obtain the dilaton dependence of these results by substituting into them the expression, eq. (1.2), for S_r in terms of S. The usual situation where $k_1 = k_2$ then implies $L_1 = L_2 \equiv L = \exp\left[-4\pi^2 kS\right]$. As for the previous example, we expect general results for global supersymmetry to preclude spontaneous supersymmetry breaking when the massless modes are integrated out, permitting us to analyze the theory's flat directions using only the Wilson superpotential. The resulting low-energy dilaton potential of this model is moderately more complicated than for the $SU(2) \times SU(2)$ theory.

Even though the superpotential does not vanish, flat directions along which S varies are easy to find. Recall that the superpotential, eq. (4.8), has the generic form:

$$W(M,S) = (\text{constant})\;\eta\; f(\xi)\,, \tag{4.19}$$

with $\xi \propto M\; e^{4\pi^2 kS/3}$ and $\eta \equiv (L_1^2/L_2)^{1/3} = e^{-4\pi^2 kS/3}$. Here $f(\xi) = (4 - X^3)X$ does not satisfy $f'(\xi) = 0$ for any finite ξ, but $f(\xi)$ is proportional to $1/\xi$ as $\xi \to \infty$.

As discussed previously, this superpotential is extremized by $\xi \to \infty$, for *any* value of η. In our previous discussions we imagined ξ being driven to ∞ by relaxing M with L_1 and L_2 fixed. Now we can do so using both M and S, so long as the combination $\xi \to \infty$. This flat direction is one along which S is free to vary.

Notice also that the gauge coupling function, τ, depends only on ξ and not separately on S. As ξ moves to infinity to minimize the scalar potential, the gauge coupling function $\tau(\xi)$ is itself driven to vanishing coupling: $\tau \to i\infty$. In this limit the low-energy $U(1)$ gauge interactions have no effect on the dilaton scalar potential. As in our previous example we are led to a degenerate, supersymmetric vacuum along which the dilaton is free to vary even after strongly-coupled, nonabelian gauge interactions are integrated out.

5. Conclusions

In this paper we have analyzed in some detail the low-energy properties of a class of $N = 1$ supersymmetric gauge theories having gauge group $SU(N_1) \times SU(N_2)$ and matter content $(\mathbf{N_1}, \mathbf{N_2}) \oplus (\overline{\mathbf{N_1}}, \overline{\mathbf{N_1}})$, with a particular eye to the dilaton scalar potential which these models predict. We have obtained the following results:

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1. We have analyzed the phase diagram of these models as functions of the free parameters, which are the quark masses, m, as well as the two gauge couplings and vacuum angles, g_1, g_2, Θ_1 and Θ_2. For m nonzero we have argued the theory to be in a confining phase, for which low-energy excitations above the confining ground state are separated from zero energy by a nonzero gap. When m is zero there are semiclassical flat directions along which the gauge group is generically broken to several $U(1)$ factors. We expect a Coulomb phase to exist along these flat directions. At special points along these flat directions it is also possible to have larger unbroken gauge symmetries, for which other phases are possible. When $N_1 \geq N_2 + 2$ we expect another phase transition as the $SU(N_1)$ gauge coupling, g_1, is turned off and the $SU(N_2)$ gauge coupling, g_2, is turned on. This expectation is based on the qualitative change in low-energy degrees of freedom which must happen as one moves from supersymmetric QCD with N_1 colours and N_2 flavours to supersymmetric QCD with N_2 colours and N_1 flavours.

2. We have found the explicit superpotentials, W, for the quantum effective action, in the confining phase of these models, a result which was previously unknown. In this phase this superpotential quite generally has the form, eq. (1.4), of a sum of exponentials which vanish as $\Re S \to \infty$, once all fields but the dilaton have been eliminated. This phase therefore always suffers from the usual runaway-dilaton problem.

   For the $SU(2) \times SU(2)$ model in particular, the model's confinement phase is subject to a nontrivial quantum constraint. We expect the same to be true for the $SU(N) \times SU(N)$ models more generally.

3. We have stated the symmetry conditions which constrain the superpotential of the Wilson action for these models. For $SU(N) \times SU(N)$ models this superpotential must vanish identically. For other gauge groups we have shown how this superpotential is related to the roots of an algebraic equation, which we cannot solve in the general case. For the particular case of the $SU(4) \times SU(2)$ model, the algebraic equation is cubic, and we find its solutions in some detail (in the phase whose low-energy spectrum is described by mesonic variables, which applies for sufficiently small g_2).

   Although the Wilson superpotentials are in general difficult to explicitly construct, we propose that the superpotential found by 'integrating in' the gaugino fields, U_r, has a simple form for the general case.

4. For the $SU(2) \times SU(2)$ and $SU(4) \times SU(2)$ models the Coulomb phase involves a single $U(1)$ gauge multiplet, and we exhibit the gauge coupling function for this multiplet explictly in terms of the modulus of an elliptic curve. Our result in the $SU(2) \times SU(2)$ case agrees with those obtained by earlier workers. In both cases our proposed coupling functions pass many nontrivial consistency checks, such

as predicting physically-reasonable β-functions, which are without singularities away from $\Im\tau = 0$, and which reproduce the known weak coupling limits. The instanton contributions to this β-function are absolute predictions of the proposed coupling function.

5. We find the dilaton superpotential for many of these models to have flat directions which survive the integration over the strongly-coupled nonabelian gauge interactions. For $SU(N) \times SU(N)$ models this is connected to the absence of a low-energy Wilson superpotential. For the $SU(4) \times SU(2)$ model the dilaton-dependent flat direction is present even though the Wilson superpotential does not identically vanish.

6. In analyzing these models we found that special care is necessary when choosing how to parameterize the system's moduli. In particular, when the moduli correspond to $D^\flat$ directions in the semiclassical limit, there are general arguments which permit these moduli to be parameterized by holomorphic gauge invariants. We have found that not all choices for these invariants give the same predictions for the low-energy physics.

   In particular, there are two natural choices for holomorphic gauge invariants that are constructed from the 'meson matrix' ${\mathcal{M}_\alpha}^\beta = Q_{a\alpha}\,\widetilde{Q}^{a\beta}$. The most widely-used invariants of this sort in the literature are the traces: $M_p = \mathrm{Tr}\,(\mathcal{M}^p)$. An alternative choice instead uses the eigenvalues, λ_p, of ${\mathcal{M}_\alpha}^\beta$. The low-energy superpotential, W, we have encountered in this paper differ in their implications, depending on whether they are expressed in terms of the M_p or the λ_p. They differ because the Jacobian of the transformation between these two sets of variables is singular along the stationary points of W. We argue that it is the λ_p which carry the correct physical implications for the analysis of interest in this paper.

Finally, it may be of interest to recover these results by using various D-brane and M-brane configurations, such as those used in ref. [27], for product group models, and those of the related technique of ref. [28].

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